MATH 53H - Solutions to Problem Set II

1. It is clear that $\ker(A-\lambda I)^v \subseteq \ker(A-\lambda I)^k$ for $k \geq v$. Thus in order to prove equality, it suffices to show the reverse inclusion. Using the fact that $gcd((x-\lambda)^k, p_A(x)) = (x-\lambda)^v$ and the Euclidean algorithm, there exist polynomials f, g such that

$$f(x)(x-\lambda)^k + g(x)p_A(x) = (x-\lambda)^v$$

Therefore, since $p_A(A) = 0$ by Cayley-Hamilton, we get $f(A)(A-\lambda)^k = (A-\lambda)^v$, which in turn clearly implies that $\ker(A-\lambda I)^k \subseteq \ker(A-\lambda I)^v$. So we are done.

2. (i) We have the following:

Lemma For $B, C \in M_{n\times n}(\mathbb{C})$, $\mathrm{tr}(BC) = \mathrm{tr}(CB)$.
Proof It is not hard to check that

$$\mathrm{tr}(BC) = \sum_{1\leq i,j\leq n} B_{ij}C_{ji} = \mathrm{tr}(CB) \qquad \square$$

Applying the lemma to $B = S^{-1}A, C = S$ we get

$$\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(SS^{-1}A) = \mathrm{tr}(A)$$

(ii) Write $A = L + N$, where L is diagonalizable, N is nilpotent and L, N commute. Then $L = P^{-1}DP$ where D is a diagonal matrix with entries the eigenvalues of A $\lambda_1, \dots, \lambda_n$. We have then $\exp(tL) = P^{-1}\exp(tD)P \Rightarrow$

$$\det(\exp(tL)) = \det(\exp(tD)) = e^{t(\lambda_1+\dots+\lambda_n)} = e^{t\,\mathrm{tr}(A)}$$

where we have used that $\exp(tD)$ is diagonal with entries $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ and the fact that the trace of a matrix equals the sum of its eigenvalues.
Since N is nilpotent, hence $N^n = 0$, $\exp(tN)$ is defined by a finite sum and therefore its entries are polynomials in t. Hence $f(t) = \det(\exp(tN))$ is a polynomial in t. Now f satisfies $f(t+s) = \det(\exp((t+s)N)) = \det(\exp(tN)\exp(sN)) = \det(\exp(tN))\det(\exp(tN)) = f(t)f(s)$ and thus for $t = s$ we have $f(2t) = f(t)^2$. Considering the degree of f this is only possible when f is constant. Hence $f(t) = f(0) = \det(\exp(0)) = \det(I) = 1$, which shows that

$$\det(\exp(tN)) = 1 \ \forall\ t$$

Combining the above, we obtain

$$\det(\exp(tA)) = \det(\exp(t(L+N))) = \det(\exp(tL)\exp(tN)) =$$

$$= \det(\exp(tL))\det(\exp(tN)) = e^{t\,\mathrm{tr}(A)}$$

3. Suppose first that A is diagonalizable. Then there is a basis of eigenvectors of A for $\mathbb{C}^n$. For any v in this basis, we have $(A-\lambda_i)v = 0$ for some i, and hence $f(A)v = 0$. Thus $f(A) = 0$ since it maps all basis vectors to zero.

Conversely suppose that $f(A) = 0$. Let v_i denote the multiplicity of λ_i. Then since $gcd(f(x),(x-\lambda_i)^{v_i}) = x - \lambda_i$, there exist polynomials p, q such that

$$p(x)f(x) + q(x)(x-\lambda_i)^{v_i} = x - \lambda_i$$

Plugging in A we get $q(A)(A-\lambda_i I)^{v_i} = A - \lambda_i I$. As in Exercise 1 this gives

$$\ker(A-\lambda_i I)^{v_i} = \ker(A-\lambda_i I)$$

i.e. each generalized eigenspace is actually an eigenspace. Therefore there is a basis of eigenvectors of A for $\mathbb{C}^n$ and A is diagonalizable.

4. **(i)** We have $p_A(\lambda) = (\lambda-1)(\lambda+2)^3$, hence the eigenvalues are 1 and -2 with corresponding multiplicities 1 and 3. The corresponding eigenspaces are $\ker(A-I)$ and $\ker(A+2I)^3$ with bases $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

(ii) It is clear from the given that $\mathbb{C}^4$ is the direct sum of the two generalized eigenspaces.

(iii) Since A is upper triangular, we have $L = D = \begin{pmatrix}1&0&0&0\\0&-2&0&0\\0&0&-2&0\\0&0&0&-2\end{pmatrix}$ and $S = I$.

(iv) We have $N = A - L = \begin{pmatrix}0&0&0&0\\0&0&1&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}$. We can check that $NL = LN$

and $N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $N^3 = 0$.

(v) We have

$$\exp(tA) = \exp(tL)\exp(tN) = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix} (I + tN + \frac{t^2N^2}{2}) =$$

$$= \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-2t} & 0 & 0 \\ 0 & 0 & e^{-2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & -t + \frac{t^2}{2} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} & (-t + \frac{t^2}{2})e^{-2t} \\ 0 & 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix}$$

5. (i) We have $p_A(\lambda) = (\lambda + i)^2(\lambda - i)^2$, hence the eigenvalue are i and $-i$ with corresponding multiplicities 2 and 2. The corresponding eigenspaces are $\ker(A + iI)^2$ and $\ker(A - iI)^2$ with bases $\left\{ v_1 = \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ i \\ 1 \end{pmatrix} \right\}$ and $\left\{ v_3 = \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ -i \\ 1 \end{pmatrix} \right\}$.

(ii) It is not hard to see that v_1, v_2, v_3, v_4 are linearly independent (they are pairwise orthogonal), hence give a basis for $\mathbb{C}^4$, which is therefore the direct sum of the two eigenspaces.

(iii) By the definition of L we get

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -\frac{i}{2}v_1 + \frac{i}{2}v_3 \Rightarrow L \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2}v_1 - \frac{1}{2}v_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

This gives us the first column of L. Similarly we obtain

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The v_i's give the columns of S and thus

$$S = \begin{pmatrix} i & 0 & -i & 0 \\ 1 & 0 & 1 & 0 \\ 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

We may find that $D = S^{-1}LS = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$.

(iv) We have $N = A - L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. We can verify that $NL = LN$ and $N^2 = 0 \Rightarrow N^4 = 0$.

(v) As in the previous exercise, we have

$$\exp(tA) = \exp(tL)\exp(tN) = S\exp(tD)S^{-1}(I + tN) =$$

$$= S \begin{pmatrix} e^{-it} & 0 & 0 & 0 \\ 0 & e^{-it} & 0 & 0 \\ 0 & 0 & e^{it} & 0 \\ 0 & 0 & 0 & e^{it} \end{pmatrix} S^{-1} \begin{pmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t & \sin t & t\cos t & t\sin t \\ -\sin t & \cos t & -t\sin t & t\cos t \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}$$